

CONVERGENCE OF RIEMANNIAN SURFACES AND CONVERGENCE OF THE BERGMAN KERNEL

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ABSTRACT. Let $\{M_j\}$ be a sequence of complete Riemannian surfaces which converges in the sense of Cheeger-Gromov to a complete Riemannian surface M . We study the convergence of the Bergman kernel K_{M_j} of M_j by using isoperimetric inequalities.

KEYWORDS: Cheeger-Gromov convergence, Bergman kernel, isoperimetric inequality.

1. INTRODUCTION

Let M be an orientable surface, i.e., an orientable differentiable 2-manifold. By means of patching up together local metrics through a partition of unity, we see that M admits many Riemannian metrics. Let

$$ds^2 = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2$$

where $EG - F^2 > 0$, $E > 0$, be a (smooth) Riemannian metric defined in local coordinates (x, y) of M . By *isothermal parameters* we mean local coordinates (ξ, ζ) with $\xi = \xi(x, y)$, $\zeta = \zeta(x, y)$, such that

$$ds^2 = \lambda(\xi, \zeta)(d\xi^2 + d\zeta^2), \quad \lambda(\xi, \zeta) > 0.$$

Such isothermal parameters are known to exist by the famous Korn-Lichtenstein theorem, which goes back to Gauss. With respect to local coordinates $z = \xi + \zeta i$, M becomes a complex manifold. This observation is significant since the complex structure of a given surface is often unknown, whereas the Riemannian metric can be analyzed by means from Riemannian geometry.

In this paper, we attempt to understand stability properties of complex analytic objects for a sequence of Riemannian surfaces which converges in the following sense:

Definition 1.1 (cf. [21], see also [23], [39]). *A sequence $\{(M_j, ds_j^2)\}$ of complete Riemannian manifolds is said to converge in the sense of Cheeger-Gromov to a complete Riemannian manifold (M, ds^2) if there exist*

- (1) *a sequence of points $p_j \in M_j$ and a point $p \in M$;*
- (2) *a sequence of precompact open sets $\Omega_j \subset M_j$ exhausting M_j , with $p \in \Omega_j$ for each j ;*
- (3) *a sequence of smooth maps $\phi_j : \Omega_j \rightarrow M_j$ which are diffeomorphic onto their image and satisfy $\phi_j(p) = p_j$;*

such that $\phi_j^(ds_j^2) \rightarrow ds^2$ in the sense that for all compact subsets $E \subset M$, the tensor $\phi_j^*(ds_j^2) - ds^2$ and its covariant derivatives of all orders (with respect to any fixed background connection) converge uniformly to zero on E .*

More precisely, we are interested in the following

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Problem 1. Let $\{(M_j, ds_j^2)\}$ be a sequence of complete Riemannian surfaces which converges in the sense of Cheeger-Gromov to a complete Riemannian surface (M, ds^2) . Let K_{M_j} (resp. K_M) be the Bergman kernel of M_j (resp. M), with respect to the corresponding complex structure. When does K_{M_j} converge to K_M in the sense that for all compact sets $E \subset M$, the tensors $\phi_j^*(K_{M_j}) - K_M$ and their covariant derivatives of all orders (with respect to any fixed background connection) converge uniformly to zero on E ?

Here the Bergman kernel is the reproducing kernel of the Hilbert space of square-integrable holomorphic differentials, which is a classical conformal invariant. Since there are plenty of convergent sequence of Riemannian surfaces whose Bergman kernels do not converge (see §10, Remark 1), so we have to find reasonable sufficient conditions. A popular global property in geometric analysis is so-called isoperimetric inequalities which we recall as follows. Let M be a noncompact complete Riemannian manifold. For each $1 \leq \nu \leq \infty$, the ν -dimensional isoperimetric constant $I_\nu(M)$ of M is defined by

$$I_\nu(M) = \inf |\partial\Omega|/|\Omega|^{1-1/\nu}$$

where the infimum is taken over all precompact domains $\Omega \subset M$ with smooth boundaries, and $|\cdot|$ stands for the volume. If $I_\nu(M) > 0$, then M satisfies isoperimetric inequalities $|\partial\Omega| \geq I_\nu(M)|\Omega|^{1-1/\nu}$ for all Ω . In case that M is compact, we have to adjust the definition of $I_\nu(M)$ as follows

$$I_\nu(M) = \inf \frac{|S|}{\min\{|\Omega_1|^{1-1/\nu}, |\Omega_2|^{1-1/\nu}\}}$$

where the infimum is taken over all compact smooth hypersurface S of M that divide M into two disjoint open subsets Ω_1, Ω_2 of M . The number $I_\infty(M)$ is also called Cheeger's constant in the literature.

Our main result is stated as follows.

Theorem 1.1. Let $\{(M_j, ds_j^2)\}$ be a sequence of complete Riemannian surfaces which converges in the sense of Cheeger-Gromov to a noncompact complete Riemannian surface (M, ds^2) . Suppose one of the following conditions is verified:

- (1) $\inf_j I_\infty(M_j) > 0$, where M_j can be compact or noncompact.
- (2) $\inf_j I_\nu(M_j) > 0$ for some $2 < \nu < \infty$, where M_j is noncompact.
- (3) $\inf_j I_2(M_j)|M_j|^{-1/2} > 0$, where M_j is compact.

Then K_{M_j} converges to K_M .

Remark. The case when M is compact is not very interesting since M_j would be diffeomorphic onto M for all sufficiently large j . Thus the classical theory on deformation of complex structures applies (compare [30]).

The idea of using the length-area method goes back to Beurling and Ahlfors, which plays an important role in the study of complex analysis on noncompact Riemannian surfaces (see e.g., [2], Chapter IV).

Condition (1) or (3) of Theorem 1.1 can be replaced by the weaker condition $\inf_j \lambda_1(M_j) > 0$, where λ_1 is the infimum of the spectrum of the Laplacian. However, condition (2) does not yield $\inf_j \lambda_1(M_j) > 0$. Thus it is reasonable to consider the case when $\lambda_1(M_j)$ degenerates, even the convergence of Riemannian surfaces is quite special.

Theorem 1.2. Let $\{(M_j, ds_j^2)\}$ be a sequence of complete Riemannian surfaces and (M, ds^2) a complete Riemannian surface. Suppose that there exists a sequence of geodesic balls $B_{R_j}(p)$ in M with p fixed and $R_j \rightarrow \infty$, such that

- (1) $B_{R_j}(p) \subset M_j$ for all j ;
- (2) $ds_j^2 = ds^2$ on $B_{R_j}(p)$;
- (3) $\inf_j \lambda_1(M_j)R_j^2 > 0$.

Then K_{M_j} converges to K_M .

When M is a \mathbb{Z} covering of a compact Riemannian surface with genus ≥ 2 , we may construct a sequence $\{M_j\}$ of compact Riemannian surfaces which converges to M as Theorem 1.2, whereas Theorem 1.1 does not apply (see §10, Remark 6).

The paper is organized as follows. In §2 we recall necessary background from geometric analysis. In §3 we review basic properties of isoperimetric inequalities. In §4 we estimate the Green function by using isoperimetric inequalities. Sections 5,6,7,8 are devoted to the proof of Theorem 1.1. In §9 we prove Theorem 1.2. In §10 we present a number of remarks.

2. BASIC FACTS FROM GEOMETRIC ANALYSIS

Let $ds^2 = g_{ij}dx^i dx^j$ be a Riemannian metric on M . The Laplace operator is defined by

$$\Delta = g^{-1/2} \frac{\partial}{\partial x^i} \left(g^{1/2} g^{ij} \frac{\partial}{\partial x^j} \right)$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$. The gradient ∇ acts on a function u by

$$(\nabla u)^i = g^{ij} \frac{\partial u}{\partial x^j}.$$

Green's formula asserts that for any precompact domain $\Omega \subset M$ with a C^1 -smooth boundary,

$$\int_{\Omega} v \Delta u = \int_{\partial\Omega} v \frac{\partial u}{\partial \vec{n}} - \int_{\Omega} \nabla v \nabla u$$

for all $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, where \vec{n} denotes the outward unit normal vector field on $\partial\Omega$.

The heat kernel $p(t, x, y)$ of M is the smallest positive fundamental solution to the heat equation

$$\partial u / \partial t = \Delta u.$$

More precisely, it is given by

$$p(t, x, y) = \lim_{j \rightarrow \infty} p_j(t, x, y)$$

where $p_j(t, x, y)$ is the Green function for the Dirichlet problem for the heat equation on the precompact open subset Ω_j , $j = 1, 2, \dots$, which exhausts M (see e.g. [19]). Some basic properties are as follows.

- (1) $p(t, x, y) = p(t, y, x)$.
- (2) $p(t, \cdot, y) \rightarrow \delta_y$ as $t \rightarrow 0+$, where δ_y denotes the Dirac distribution.
- (3) The semigroup property: for all $t, s > 0$ and $x, y \in M$,

$$p(t + s, x, y) = \int_M p(t, x, z) p(s, y, z) dV_z.$$

- (4) $\int_M p(t, x, y) dV_y \leq 1$.

A positive increasing function κ on $(0, \infty)$ is called *regular* if there are numbers $A \geq 1$ and $\beta > 1$ such that

$$(2.1) \quad \frac{\kappa(\beta s)}{\kappa(s)} \leq A \frac{\kappa(\beta t)}{\kappa(t)}, \quad \forall 0 < s < t.$$

Grigor'yan made a deep observation that *off-diagonal* estimates of the heat kernel may be deduced from (easier) *on-diagonal* estimates as follows.

Theorem 2.1 (cf. [18]). *Let x, y be two points in M such that for all $t > 0$*

$$p(t, x, x) \leq 1/\kappa_1(t), \quad p(t, y, y) \leq 1/\kappa_2(t)$$

where κ_1, κ_2 are two regular functions. Then for any number $\alpha < 1$ and all $t > 0$

$$(2.2) \quad p(t, x, y) \leq \frac{4A}{\sqrt{\kappa_1(\delta t)\kappa_2(\delta t)}} \exp\left(-\frac{\alpha d^2(x, y)}{4t}\right)$$

where $\delta = \delta(\beta, \alpha)$ and A, β are the constants from (2.1).

Finally, let M be a Riemannian n -manifold with Ricci curvature $\geq -(n-1)b^2$ where $b \geq 0$. Let $B_r(x)$ denote the geodesic ball with center x and radius r . Suppose $\overline{B_r(x)} \subset M$. Then we have

(1) Harnack's inequality (cf. [36]): For any positive harmonic function u on $B_r(x)$,

$$(2.3) \quad \sup_{B_{r/2}(x)} u \leq e^{\text{const}_n(1+br)} \inf_{B_{r/2}(x)} u.$$

(2) The sub-mean-value inequality (cf. [32]): For any positive subharmonic function u on $B_r(x)$,

$$(2.4) \quad \sup_{B_{r/2}(x)} u^2 \leq e^{\text{const}_n(1+br)} |B_r(x)|^{-1} \int_{B_r(x)} u^2.$$

For further knowledge on geometric analysis, one may consult the book of Schoen-Yau [36] and survey articles of Grigor'yan [19], [20].

3. ISOPERIMETRIC INEQUALITIES

We follow closely the books of Chavel [8], [7]. Let M be a noncompact complete Riemannian n -manifold. Let \mathcal{F} denote the set of precompact domains in M with smooth boundaries. For each $1 \leq \nu \leq \infty$, the ν -dimensional isoperimetric constant $I_\nu(M)$ of M is defined by

$$I_\nu(M) = \inf_{\Omega \in \mathcal{F}} |\partial\Omega|/|\Omega|^{1-1/\nu}.$$

Similarly, we may define for each $\nu > 1$ the ν -dimensional Sobolev constant

$$(3.1) \quad S_\nu(M) = \inf \{ \|\nabla u\|_1 / \|u\|_{\nu/(\nu-1)} : u \in C_0^\infty(M) \}$$

where $C_0^\infty(M)$ denotes the set of smooth functions with compact supports in M and $\|\cdot\|_p$ stands for the standard L^p -norm. The famous Federer-Fleming-Maz'ya inequality yields

$$(3.2) \quad I_\nu(M) = S_\nu(M)$$

for all $\nu \in (1, \infty]$. For each $\phi \in C_0^\infty(M)$, we put $u = |\phi|^{2(\nu-1)/(\nu-2)}$ for some $\nu > 2$. By (3.1), (3.2) and the Cauchy-Schwarz inequality, we immediately get the following L^2 Sobolev inequality

$$(3.3) \quad \|\nabla \phi\|_2 \geq \frac{\nu-2}{2(\nu-1)} I_\nu(M) \|\phi\|_{2\nu/(\nu-2)}.$$

Thanks to the Hölder inequality, we have

$$\int_M \phi^{2+4/\nu} = \int_M \phi^2 \cdot \phi^{4/\nu} \leq \left(\int_M \phi^{2\nu/(\nu-2)} \right)^{(\nu-2)/\nu} \left(\int_M \phi^2 \right)^{2/\nu}$$

and

$$\int_M \phi^2 = \int_M \phi^{4/(\nu+4)} \cdot \phi^{(2\nu+4)/(\nu+4)} \leq \left(\int_M |\phi| \right)^{4/(\nu+4)} \left(\int_M \phi^{2+4/\nu} \right)^{\nu/(\nu+4)}.$$

Together with (3.3), we obtain Nash's inequality

$$(3.4) \quad \|\phi\|_2^{2+4/\nu} \leq \left(\frac{\nu-2}{2(\nu-1)} I_\nu(M) \right)^{-2} \|\nabla \phi\|_2^2 \cdot \|\phi\|_1^{4/\nu}, \quad \forall \nu > 2.$$

A central property of I_ν is that it behaves well under *rough isometries*. Following Kanai [27], we call a map $\Phi : M_1 \rightarrow M_2$ between two Riemannian manifolds M_1 and M_2 a *rough isometry* if there are constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1}d_1(x, y) - b \leq d_2(\Phi(x), \Phi(y)) \leq ad_1(x, y) + b$$

for all $x, y \in M_1$, and Φ is *r-full* for some $r > 0$, i.e.,

$$\bigcup_{x \in M_1} B_r(\Phi(x)) = M_2.$$

A complete Riemannian manifold M has *bounded geometry* if the Ricci curvature is bounded below by a constant, and the injectivity radius $\text{inj}(M)$ of M is positive.

Theorem 3.1 (cf. [27]). *Let M_1, M_2 be complete Riemannian manifolds with bounded geometries such that they are roughly isometric to each other. Let*

$$\nu \geq \max\{\dim M_1, \dim M_2\}.$$

Then $I_\nu(M_1) > 0$ if and only if $I_\nu(M_2) > 0$.

Below we collect some examples concerning positive isoperimetric constants (cf. [19], § 7):

- (1) Any Cartan-Hadamard n -manifold has $I_n(M) > 0$.
- (2) Any Cartan-Hadamard manifold M with sectional curvature $\leq -b^2$ ($b > 0$) has $I_\infty(M) > 0$.
- (3) Any n -dimensional minimal submanifold M in \mathbb{R}^N has $I_n(M) > 0$. Note that any complex submanifold in \mathbb{C}^n is minimal.

A useful isoperimetric inequality is given by Coulhon and Saloff-Coste (cf. [12], Theorem 4, see also [20], Theorem 11.3) as follows. Let M be a noncompact regular covering of a compact manifold M_0 . Put

$$V(r) := |B_r(x_0)|,$$

where x_0 is some (fixed) point in M . For some (large) constant $C > 0$, the isoperimetric inequality

$$(3.5) \quad |\partial\Omega| \geq \frac{|\Omega|}{CV^{-1}(C|\Omega|)}$$

holds for all precompact domains $\Omega \subset M$ with smooth boundaries and $|\Omega| \geq \text{const.} > 0$. Here V^{-1} is the inverse function of V . In particular, any \mathbb{Z}^m ($m \geq 2$) covering of a compact Riemannian manifold has $I_m > 0$.

There is also a beautiful example from *hyperbolic* geometry. Let \mathbb{H}^n be the hyperbolic space. A hyperbolic manifold is given by $M = \mathbb{H}^n/\Gamma$ where Γ is a free, discrete group of

hyperbolic isometries. The *critical exponent* $\delta(\Gamma)$ of Poincaré series is defined by

$$\delta(\Gamma) = \inf \left\{ s \geq 0 : \sum_{\sigma \in \Gamma} \exp(-sd(x, \sigma(y))) < \infty \right\}$$

for some/any $x, y \in \mathbb{H}^n$, where d denotes the hyperbolic distance. It is well-known that $\delta(\Gamma) \leq n - 1$. Let $\lambda_1(M)$ be the fundamental tone of M , i.e., the infimum of the spectrum of $-\Delta$. The quantities $I_\infty(M)$, $\lambda_1(M)$ and $\delta(\Gamma)$ are related through the following

- (1) Cheeger's inequality (cf. [10]): $\lambda_1(M) \geq I_\infty(M)^2/4$ (actually holds arbitrary complete manifolds);
- (2) Sullivan's theorem (cf. [37]): $\lambda_1(M) = (n-1)^2/4$ if $\delta(\Gamma) \leq (n-1)/2$, and $\lambda_1(M) = \delta(\Gamma)(n-1-\delta(\Gamma))$ otherwise;
- (3) Buser's inequality (cf. [5]): $\lambda_1(M) \leq \text{const}_n I_\infty(M)$ (actually holds for arbitrary noncompact complete manifolds with Ricci curvature ≥ -1).

It follows immediately that

$$(3.6) \quad \delta(\Gamma) < n - 1 \iff I_\infty(M) > 0.$$

In particular, most hyperbolic Riemannian surfaces have $I_\infty(M) > 0$.

Based on Theorem 3.1 and the examples above, we may construct many complete Riemannian *surfaces* with $I_\nu > 0$ for some $\nu > 2$. For instance, a 2-dimensional jungle gym in \mathbb{R}^n ($n > 2$) has $I_n > 0$, whereas a 2-dimensional jungle gym in a Cartan-Hadamard manifold with sectional curvature $\leq -b^2$ ($b > 0$) has $I_\infty > 0$.

4. ESTIMATES OF THE GREEN FUNCTION

Let M be a noncompact complete Riemannian n -manifold. Let $\Omega \subset M$ be an open set and U be a precompact open set in Ω . The capacity $\text{cap}(U, \Omega)$ is defined as follows

$$\text{cap}(U, \Omega) = \inf \int_{\Omega} |\nabla \phi|^2$$

where the infimum is taken over all locally Lipschitz functions on M with compact supports in $\overline{\Omega}$ such that $0 \leq \phi \leq 1$ and $\phi|_{\overline{U}} = 1$. Let g_Ω denote the (positive) Green function of Ω . There is a useful link between the capacity and the Green function as follows

$$(4.1) \quad \inf_{\partial U} g_\Omega(\cdot, y) \leq \text{cap}(U, \Omega)^{-1} \leq \sup_{\partial U} g_\Omega(\cdot, y), \quad \forall y \in U$$

(cf. [20], Proposition 4.1).

Now fix $o \in M$ and let $B_R := B_R(o)$ with $R > 1$. Put

$$\varepsilon_R := \min \left\{ \inf_{x \in B_{R+1}} \text{inj}(M, x), 1/2 \right\},$$

where $\text{inj}(M, x)$ denotes the injectivity radius at x . We give first a rough lower bound for the Green function g_M of M as follows.

Proposition 4.1. *Suppose $\text{Ricci}(M) \geq -(n-1)b^2$ ($b \geq 0$) on B_{R+1} . Then*

$$(4.2) \quad g_M(x, o) \geq \frac{1}{8} |B_{R+1}|^{-1} \exp\{-\text{const}_n (1 + b\varepsilon_R) \varepsilon_R^{-n} |B_{R+1}|\}.$$

for all $x \in B_R$.

Proof. Take $\{x_1, \dots, x_m\} \subset \partial B_R$ such that $B_{\frac{1}{2}\varepsilon_R}(x_1), \dots, B_{\frac{1}{2}\varepsilon_R}(x_m)$ do not overlap and $B_{\varepsilon_R}(x_1), \dots, B_{\varepsilon_R}(x_m)$ cover ∂B_R . By a well-known theorem of Croke [13] we have

$$\left| B_{\frac{1}{2}\varepsilon_R}(x_k) \right| \geq \text{const}_n \varepsilon_R^n$$

for all k . Thus

$$|B_{R+1}| \geq \sum_k \left| B_{\frac{1}{2}\varepsilon_R}(x_k) \right| \geq m \text{const}_n \varepsilon_R^n,$$

i.e.,

$$m \leq (\text{const}_n \varepsilon_R^n)^{-1} |B_{R+1}|.$$

Since $g_{B_{R+1}}(\cdot, o)$ is harmonic on $B_{R+1} \setminus \{o\}$, it follows from Harnack's inequality (2.3) that

$$\sup_{B_{\varepsilon_R}(x_k)} g_{B_{R+1}}(\cdot, o) \leq e^{\text{const}_n(1+b\varepsilon_R)} \inf_{B_{\varepsilon_R}(x_k)} g_{B_{R+1}}(\cdot, o)$$

for all k . Since $B_{\varepsilon_R}(x_1), \dots, B_{\varepsilon_R}(x_m)$ cover ∂B_R , so we have

$$\begin{aligned} \sup_{\partial B_R} g_{B_{R+1}}(\cdot, o) &\leq e^{\text{const}_n(1+b\varepsilon_R)m} \inf_{\partial B_R} g_{B_{R+1}}(\cdot, o) \\ (4.3) \quad &\leq e^{\text{const}_n(1+b\varepsilon_R)\varepsilon_R^{-n}|B_{R+1}|} \inf_{\partial B_R} g_{B_{R+1}}(\cdot, o). \end{aligned}$$

By virtue of Theorem 7.1 in [20], we have

$$\begin{aligned} \text{cap}(B_R, B_{R+1})^{-1} &\geq \frac{1}{2} \int_R^{R+1} \frac{(t-R)dt}{|B_t| - |B_R|} \\ &\geq \frac{1}{2} \int_{R+1/2}^{R+1} \frac{(t-R)dt}{|B_t| - |B_R|} \\ &\geq \frac{1}{8} |B_{R+1}|^{-1}. \end{aligned}$$

Together with (4.1) and (4.3), we get

$$\inf_{\partial B_R} g_M(\cdot, o) \geq \inf_{\partial B_R} g_{B_{R+1}}(\cdot, o) \geq \frac{1}{8} |B_{R+1}|^{-1} \exp\{-\text{const}_n(1+b\varepsilon_R)\varepsilon_R^{-n}|B_{R+1}|\}.$$

The assertion follows immediately from the maximal principle. \square

In what follows in this section we always assume that M is a noncompact complete Riemannian manifold with $I_\nu(M) > 0$ for some $2 < \nu < \infty$. We have the following (probably optimal) upper bound for the Green function.

Proposition 4.2 (cf. [9]). *We have*

$$(4.4) \quad g_M(x, y) \leq \text{const}_\nu I_\nu(M)^\nu d(x, y)^{2-\nu}$$

for all $x, y \in M$.

In order to make the paper self-contained, we include the proof here. The key point is to obtain the following Gaussian upper bound for the heat kernel.

Theorem 4.3 (cf. [9] or [19]). *For any $\alpha < 1$,*

$$(4.5) \quad p(t, x, y) \leq \text{const}_{\nu, \alpha} I_\nu(M)^\nu t^{-\nu/2} \exp\left(-\frac{\alpha d^2(x, y)}{4t}\right)$$

for all $x, y \in M$ and $t > 0$.

Let us first observe how to derive (4.4) from (4.5). Indeed,

$$\begin{aligned} g_M(x, y) &= \int_0^\infty p(t, x, y) dt \leq \text{const}_\nu I_\nu(M)^\nu \int_0^\infty t^{-\nu/2} \exp\left(-\frac{d^2(x, y)}{5t}\right) dt \\ &\leq \text{const}_\nu I_\nu(M)^\nu d(x, y)^{2-\nu} \int_0^\infty t^{-\nu/2} \exp\left(-\frac{1}{5t}\right) dt. \end{aligned}$$

In order to verify (4.5), we need an *on-diagonal* upper bound for the heat kernel which goes back to Nash (see [19], § 6.1). By a standard exhaustion argument, it suffices to work on a precompact open set $\Omega \subset M$ with a smooth boundary. Fix $y \in \Omega$ and put $u(t, x) := p_\Omega(t, x, y)$ and

$$J(t) = \int_\Omega u^2(t, x) dV_x.$$

Note that

$$\begin{aligned} J'(t) &= 2 \int_\Omega uu_t = 2 \int_\Omega u \Delta u = -2 \int_\Omega |\nabla u|^2 \\ &\leq -\text{const}_\nu I_\nu(M)^{-2} \|u\|_2^{2+4/\nu} \|u\|_1^{-4/\nu} \end{aligned}$$

in view of Nash's inequality (3.4). Since $\|u\|_1 = \|p_\Omega(t, \cdot, y)\|_1 \leq 1$, we have

$$J' \leq -\text{const}_\nu I_\nu(M)^{-2} J^{1+2/\nu},$$

so that

$$J(t) \leq \text{const}_\nu I_\nu(M)^\nu t^{-\nu/2},$$

for $J(0+) = \infty$. By the semigroup property, we obtain

$$p_\Omega(t, y, y) = J(t/2) \leq \text{const}_\nu I_\nu(M)^\nu t^{-\nu/2} =: 1/\kappa_\nu(t).$$

For all $0 < s < t$, we have

$$\frac{\kappa_\nu(2s)}{\kappa_\nu(s)} = \frac{\kappa_\nu(2t)}{\kappa_\nu(t)},$$

so that κ_ν is regular with $A = 1$ and $\beta = 2$. It follows from Theorem 2.1 that for any number $\alpha < 1$,

$$p(t, x, y) \leq \frac{4}{\kappa_\nu(\delta_\alpha t)} \exp\left(-\frac{\alpha d^2(x, y)}{4t}\right)$$

holds for suitable constant $\delta_\alpha > 0$, from which inequality (4.5) immediately follows.

Remark. (1) *There are no analogous upper bounds for g_M when $I_\infty(M) > 0$ (consider the punctured disc with the Poincaré metric).*

(2) *It is interesting to note that Blocki used the classical isoperimetric inequality in \mathbb{C} (i.e., $I_2(\mathbb{C}) > 0$) to show that*

$$\log |\{g_\Omega(\cdot, y) > t\}| + 2t$$

is non-increasing in $t \in [0, \infty)$ for any $y \in \Omega \subset \mathbb{C}$ (see e.g., [4], Theorem 10.1).

5. EFFECTIVE CONVERGENCE OF THE BERGMAN KERNEL

Let (M, ds^2) be a noncompact complete Riemannian surface. Let $\lambda_1(M)$ be the infimum of the spectrum of $-\Delta$, i.e.,

$$\lambda_1(M) = \inf \left\{ \frac{\int_M |df|^2}{\int_M |f|^2} : f \in C_0^\infty(M) \setminus \{0\} \right\}.$$

Now view (M, ds^2) as a complex manifold with ds^2 given by $\lambda dw d\bar{w}$ in local holomorphic coordinates. The *complex* Laplace operator is defined by

$$\square = \lambda^{-1} \frac{\partial^2}{\partial w \partial \bar{w}} = \frac{1}{4} \Delta.$$

Let $C_0^\infty(M, \mathbb{C})$ denote the set of complex-valued smooth functions on M with compact supports. We begin with an elementary remark.

Lemma 5.1.

$$(5.1) \quad \lambda_1(M) = 4 \inf \left\{ \frac{\int_M |\partial f|^2}{\int_M |f|^2} : f \in C_0^\infty(M, \mathbb{C}) \setminus \{0\} \right\}.$$

Proof. For all $f \in C_0^\infty(M, \mathbb{C})$, we have

$$\int_M |\bar{\partial} f|^2 = \frac{\sqrt{-1}}{2} \int_M \partial \bar{f} \wedge \bar{\partial} f = -\frac{\sqrt{-1}}{2} \int_M \bar{f} \partial \bar{\partial} f = \frac{\sqrt{-1}}{2} \int_M \partial f \wedge \bar{\partial} \bar{f} = \int_M |\partial f|^2,$$

so that

$$\lambda_1(M) \int_M |f|^2 \leq \int_M |df|^2 \leq 2 \int_M |\partial f|^2 + 2 \int_M |\bar{\partial} f|^2 = 4 \int_M |\partial f|^2.$$

On the other side, we may choose a sequence $f_j \in C_0^\infty(M, \mathbb{R}) \setminus \{0\}$ such that

$$\frac{\int_M |df_j|^2}{\int_M |f_j|^2} \rightarrow \lambda_1(M).$$

Since f_j is real-valued, so we have $|df_j|^2 = 4|\partial f_j|^2$ and

$$4 \frac{\int_M |\partial f_j|^2}{\int_M |f_j|^2} \rightarrow \lambda_1(M).$$

□

Proposition 5.2. *Suppose $\lambda_1(M) > 0$. Then for each $(1, 1)$ -form v with $\|v\|_2 < \infty$, there exists a solution u of the equation $\bar{\partial} u = v$ such that*

$$\|u\|_2 \leq \frac{2}{\sqrt{\lambda_1(M)}} \|v\|_2.$$

Proof. Let $D_{(p,q)}(M)$ denote the set of smooth (p, q) -forms with compact supports in M . We introduce the following inner product

$$(f_1, f_2) = \int_M \phi_1 \bar{\phi}_2 \lambda^{-1} dV_w$$

for all $f_1 = \phi_1 dw \wedge d\bar{w}$, $f_2 = \phi_2 dw \wedge d\bar{w} \in D_{(1,1)}(M)$, where $dV_w = \frac{\sqrt{-1}}{2} dw \wedge d\bar{w}$. Let $\bar{\partial}^*$ be the formal adjoint of $\bar{\partial}$. For any $u = \psi dw \in D_{(1,0)}(M)$ and $f = \phi dw \wedge d\bar{w} \in D_{(1,1)}(M)$, we have

$$(f, \bar{\partial} u) = - \int_M \phi \frac{\partial \bar{\psi}}{\partial w} \lambda^{-1} dV_w = \int_M \frac{\partial}{\partial w} (\lambda^{-1} \phi) \bar{\psi} dV_w,$$

so that

$$\bar{\partial}^* f = \frac{\partial}{\partial w} (\lambda^{-1} \phi) dw.$$

Since $\tilde{f} := \lambda^{-1} \phi \in C_0^\infty(M, \mathbb{C})$, it follows that $\bar{\partial}^* f = \partial \tilde{f}$ and

$$(5.2) \quad \int_M |f|^2 \leq \frac{4}{\lambda_1(M)} \int_M |\bar{\partial}^* f|^2$$

in view of (5.1). The remaining argument is standard (see e.g., [24], p. 249). Given $v \in L_{(1,1)}^2(M, \mathbb{C})$, the linear functional

$$\text{Range } \bar{\partial}^* \rightarrow \mathbb{C}, \quad \bar{\partial}^* f \mapsto (f, v)$$

is well-defined and bounded by $2\lambda_1(M)^{-1/2} \|v\|_2$ in view of (5.2). Thus by Hahn-Banach's theorem and the Riesz representation theorem, there is a unique $u \in L_{(1,0)}^2(M, \mathbb{C})$ such that

$$(\bar{\partial}^* f, u) = (f, v)$$

for all $f \in D_{(1,1)}(M)$, i.e., $\bar{\partial} u = v$ holds in the sense of distributions, such that

$$\int_M |u|^2 \leq \frac{4}{\lambda_1(M)} \int_M |v|^2.$$

□

Let $\mathcal{H}(M)$ denote the Hilbert space of holomorphic differentials h on M satisfying

$$\|h\|_2^2 := \frac{\sqrt{-1}}{2} \int_M h \wedge \bar{h} < \infty.$$

Let $\{h_j\}_{j=1}^\infty$ be a complete orthonormal basis of $\mathcal{H}(M)$. The Bergman kernel K_M of M is given by

$$K_M(z) = \sum_j h_j(z) \wedge \overline{h_j(z)}, \quad \forall z \in M.$$

Proposition 5.3. *Let M be as the proposition above. Let ρ denote the distance from some fixed point $o \in M$. Let K_M and K_{B_R} denote the Bergman kernel of M and the geodesic ball $B_R := B_R(o)$ respectively. For each compact set $E \subset M$ with $o \in E$, we have*

$$(5.3) \quad \sup_{z \in E} |K_M(z) - K_{B_R}(z)| \leq \text{const.} \left(\lambda_1(M)^{-1/2} R^{-1} + \lambda_1(M)^{-1} R^{-2} \right)$$

for all $R > 2(\text{diam } E + 1)$, where the constant depends only on $\inf_{x \in E} |B_1(x)|$ and the infimum of the Gaussian curvature of ds^2 on $E_1 := \{z \in M : \text{dist}(z, E) \leq 1\}$.

Proof. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function satisfying $\chi|_{(-\infty, 1/2)} = 1$ and $\chi|_{(1, \infty)} = 0$. Fix $R > 2(\text{diam } E + 1)$ for a moment. Let h be a holomorphic differential on B_R with $\|h\|_2 \leq 1$. Put

$$v = \bar{\partial} \chi(\rho/R) \wedge h.$$

By virtue of Proposition 5.2, there is a solution of $\bar{\partial} u = v$ on M which satisfies

$$(5.4) \quad \|u\|_2^2 \leq \frac{4}{\lambda_1(M)} \|v\|_2^2 \leq \frac{4}{\lambda_1(M)} \frac{\sup |\chi'|^2}{R^2}.$$

Since $E_1 \subset B_{R/2}$, we see that u is holomorphic on E_1 , for $v = 0$ holds on $B_{R/2}$. Write

$$u = \phi dw \quad \text{and} \quad ds^2 = \lambda dw d\bar{w}$$

in local holomorphic coordinates. Then we have

$$\log |u|^2 = \log |\phi|^2 - \log \lambda,$$

so that

$$\Delta \log |u|^2 = 4 \square \log |u|^2 \geq -\frac{4}{\lambda} \frac{\partial \log \lambda}{\partial w \partial \bar{w}} \geq -2b^2$$

holds on E_1 , where $-b^2$ ($b \geq 0$) denotes the infimum of the Gaussian curvature of ds^2 on E_1 . It follows immediately that

$$\hat{u}(t, z) := b^2 t^2 + \log |u|^2$$

is subharmonic with respect to the metric $dt^2 + ds^2$ on $\mathbb{R} \times E_1$, so is $e^{\hat{u}/2}$. Applying the sub-mean-value inequality (2.4) to $e^{\hat{u}/2}$, we obtain that for any $z \in E$,

$$\begin{aligned} |u|^2(z) = e^{\hat{u}(0,z)} &\leq e^{C_0(1+b)} |B_1(0, z)|^{-1} \int_{B_1(0,z)} e^{\hat{u}} \\ &\leq e^{C_0(1+b)} \int_0^1 e^{b^2 t^2} dt \cdot |B_1(z)|^{-1} \int_{B_1(z)} |u|^2 \\ &\leq e^{C_0(1+b)} \int_0^1 e^{b^2 t^2} dt \cdot |B_1(z)|^{-1} \frac{4}{\lambda_1(M)} \frac{\sup |\chi'|^2}{R^2} \\ &\leq C \lambda_1(M)^{-1} R^{-2} \end{aligned}$$

in view of (5.4). Here C_0 is a universal constant and C is a generic constant depending only on b , and $\inf_{x \in E} |B_1(x)|$.

Put $\tilde{h} := \chi(\rho/R)h - u$. Clearly, \tilde{h} is a holomorphic differential on M which satisfies

$$\|\tilde{h}\|_2 \leq 1 + \frac{2}{\sqrt{\lambda_1(M)}} \sup |\chi'| R^{-1}$$

and

$$\left| \tilde{h}(z) \wedge \overline{\tilde{h}(z)} - h(z) \wedge \overline{h(z)} \right| = |u(z)|^2 \leq C \lambda_1(M)^{-1} R^{-2}$$

for all $z \in E$. It follows that

$$\sqrt{-1} K_M(z) \geq \sqrt{-1} \frac{\tilde{h}(z) \wedge \overline{\tilde{h}(z)}}{\|\tilde{h}\|_2^2} \geq \frac{\sqrt{-1} h(z) \wedge \overline{h(z)} - C \lambda_1(M)^{-1} R^{-2} \omega}{(1 + C \lambda_1(M)^{-1/2} R^{-1})^2}$$

where ω denotes the *Kähler form* of ds^2 . Since h can be arbitrarily chosen, we have

$$(5.5) \quad \sqrt{-1} K_M(z) - \frac{\sqrt{-1} K_{B_R}(z)}{(1 + C \lambda_1(M)^{-1/2} R^{-1})^2} \geq -C \lambda_1(M)^{-1} R^{-2} \omega.$$

Let $z_0 \in E$ be fixed for a moment. We may choose a holomorphic differential h_0 on M with unit L^2 -norm, such that

$$K_{B_R}(z_0) = h_0(z_0) \wedge \overline{h_0(z_0)}.$$

Applying (2.4) to $|h_0|^2$ in a similar way as above, we obtain $|K_{B_R}(z_0)| \leq C$. Together with (5.5), we get

$$\sqrt{-1} K_M(z_0) - \sqrt{-1} K_{B_R}(z_0) \geq -C(\lambda_1(M)^{-1/2} R^{-1} + \lambda_1(M)^{-1} R^{-2}) \omega.$$

Since $\sqrt{-1} K_M(z) \leq \sqrt{-1} K_{B_R}(z)$ holds trivially, so we conclude the proof. \square

Note that $\lambda_1(M) = 0$ holds for any compact Riemannian manifold. Thus we have to adjust the definition of $\lambda_1(M)$ as follows

$$\lambda_1(M) = \inf \frac{\int_M |df|^2}{\int_M |f|^2}$$

where the infimum is taken over all C^∞ -smooth real-valued functions f on M such that $\int_M f = 0$. Similar as (5.1), we can verify that

$$\lambda_1(M) = 4 \inf \frac{\int_M |\partial f|^2}{\int_M |f|^2}$$

where the infimum is taken over all C^∞ -smooth complex-valued functions f on M such that $\int_M f = 0$.

Proposition 5.4. *Let M be a compact Riemannian surface with $\lambda_1(M) > 0$. Let ρ denote the distance from some fixed point $o \in M$. Let E be a compact subset in M with $o \in E$. We have*

$$(5.6) \quad \sup_{z \in E} |K_M(z) - K_{B_R}(z)| \leq \text{const.} \left(\lambda_1(M)^{-1/2} R^{-1} + \lambda_1(M)^{-1} R^{-2} \right)$$

for all $R > 2(\text{diam } E + 1)$, where the constant depends only on $\inf_{x \in E} |B_1(x)|$ and the infimum of the Gaussian curvature of ds^2 on $E_1 := \{z \in M : \text{dist}(z, E) \leq 1\}$.

Proof. Let $H_{(p,q)}^2(M)$ denote the space of L^2 harmonic (p, q) -forms on M . Clearly, $H_{(1,0)}^2(M)$ coincides with the Bergman space $\mathcal{H}(M)$. Now we determine $H_{(1,1)}^2(M)$. Since

$$\bar{\partial}^* f = \partial \tilde{f}$$

where $f = \phi dw \wedge d\bar{w} \in C_{(1,1)}^\infty(M)$ and $\tilde{f} = \lambda^{-1} \phi \in C^\infty(M, \mathbb{C})$, we see that $f_0 := \lambda dw \wedge d\bar{w} \in H_{(1,1)}^2(M)$, for $\bar{\partial}^* f_0 = \partial f_0 = 0$. On the other hand, it follows from the Serre Duality Theorem that $H_{(1,1)}^2(M) \simeq H_{(0,0)}^2(M) \simeq \mathbb{C}$. Thus $H_{(1,1)}^2(M) = \mathbb{C} \cdot f_0$. Now suppose $f = \phi dw \wedge d\bar{w} \in H_{(1,1)}^2(M)^\perp \cap C_{(1,1)}^\infty(M)$, i.e., $(f, f_0) = \int_M \phi dV_w = 0$. It follows that $\tilde{f} = \lambda^{-1} \phi \in C^\infty(M, \mathbb{C})$ and $\int_M \tilde{f} = 0$, so that

$$\|\tilde{f}\|_2^2 \leq \frac{4}{\lambda_1(M)} \|\partial \tilde{f}\|_2^2,$$

i.e.,

$$\|f\|_2^2 \leq \frac{4}{\lambda_1(M)} \|\bar{\partial}^* f\|_2^2.$$

The Hodge Decomposition Theorem asserts that

$$C_{(1,0)}^\infty(M) = \mathcal{H}(M) \oplus \bar{\partial}^* C_{(1,1)}^\infty(M).$$

Thus for any $u \in \mathcal{H}(M)^\perp \cap C_{(1,0)}^\infty(M)$ we may write $u = \bar{\partial}^* f$ for some $f \in C_{(1,1)}^\infty(M)$. Without loss of generality, we may choose f such that it is orthogonal to $\text{Ker } \bar{\partial}^* = H_{(1,1)}^2(M)$. It follows that

$$(u, u) = (u, \bar{\partial}^* f) = (\bar{\partial} u, f) \leq \|\bar{\partial} u\|_2 \|f\|_2 \leq \frac{2}{\sqrt{\lambda_1(M)}} \|\bar{\partial} u\|_2 \|u\|_2,$$

i.e.,

$$\|u\|_2^2 \leq \frac{4}{\lambda_1(M)} \|\bar{\partial} u\|_2^2.$$

Let ρ, χ, h be as above. Replacing ρ by a smoothing of it, we may assume, without loss of generality, that it is C^∞ on M and $|d\rho| \leq 2$. Then we have the following orthogonal decomposition

$$\chi(\rho/R)h = \tilde{h} \oplus u$$

where $\tilde{h} \in \mathcal{H}(M)$ and $u \in \mathcal{H}(M)^\perp \cap C_{(1,0)}^\infty(M)$. Clearly, we have

$$\bar{\partial}u = \bar{\partial}\chi(\rho/R) \wedge h =: v$$

and

$$\|u\|_2^2 \leq \frac{4}{\lambda_1(M)} \|v\|_2^2.$$

The remaining argument is essentially similar as Proposition 5.3. \square

Corollary 5.5. *Let M be a complete Riemannian surface.*

(1) *If $I_\infty(M) > 0$ where M can be compact or noncompact, then*

$$(5.7) \quad \sup_{z \in E} |K_M(z) - K_{B_R}(z)| \leq C (I_\infty(M)^{-1} R^{-1} + I_\infty(M)^{-2} R^{-2}).$$

(2) *If M is compact, then*

$$(5.8) \quad \sup_{z \in E} |K_M(z) - K_{B_R}(z)| \leq C \left(\frac{\sqrt{|M|}}{I_2(M)R} + \frac{|M|}{I_2(M)^2 R^2} \right).$$

Here the constants are the same as the propositions above.

Proof. Inequality (5.7) follows from Proposition 5.3, Proposition 5.4, and Cheeger's inequality

$$\lambda_1(M) \geq \frac{1}{4} I_\infty(M)^2$$

(see [10]). Inequality (5.8) follows from Proposition 5.4 and P. Li's estimate

$$\lambda_1(M) \geq C_0 \frac{I_2(M)^2}{|M|},$$

where C_0 is a universal constant (see [31], Proposition 3). \square

Proposition 5.6. *Let M be a noncompact complete Riemannian surface with $I_\nu(M) > 0$ for some $2 < \nu < \infty$. Let $o \in M$ be fixed and let $B_R = B_R(o)$. For any compact subset $E \subset M$ with $o \in E$, we have*

$$(5.9) \quad \sup_{z \in E} |K_M(z) - K_{B_R}(z)| \leq C_1 |\log R|^{-1}$$

for all $R > C_2$, where the constants C_1, C_2 depend only on $\nu, I_\nu(M), \text{diam } E, \inf_{x \in E} |B_1(x)|, |B_{2+\text{diam } E}|, \inf_{x \in B_{2+\text{diam } E}} \text{inj}(M, x)$ and the infimum of the Gaussian curvature of ds^2 on $B_{2+\text{diam } E}$.

Proof. Let ρ, χ, h be as above. Put

$$\kappa = \chi (-\log \log(g_M(\cdot, o) + 1) + \log \log(C_\nu R^{2-\nu} + 1) + 1)$$

where $C_\nu = \text{const}_\nu I_\nu(M)^\nu$ is the constant from (4.4). Note that

$$\text{supp } \kappa \subset \{g_M(\cdot, o) \geq C_\nu R^{2-\nu}\} \subset B_R$$

in view of (4.4). Since

$$-i\partial\bar{\partial} \log g_M(\cdot, o) \geq i\partial \log g_M(\cdot, o) \wedge \bar{\partial} \log g_M(\cdot, o)$$

holds on $M \setminus \{o\}$, we infer from the L^2 estimate of Donnelly-Fefferman (see [16], [14], [3]) that there exists a solution of the equation

$$\bar{\partial}u = \bar{\partial}(\kappa h)$$

such that

$$\begin{aligned} \int_{M \setminus \{o\}} |u|^2 &\leq C_0 \int_{M \setminus \{o\}} |\bar{\partial}\kappa|_{-i\partial\bar{\partial}\log g_M(\cdot, o)}^2 |h|^2 \\ &\leq \text{const}_{\nu, I_\nu(M)} |\log R|^{-2} \end{aligned}$$

where C_0 is a universal constant. On the other side, since $g_M(\cdot, o) \geq C_3 > 0$ on $B_{1+\text{diam } E}$ in view of (4.2), so we have

$$\{\kappa = 1\} \supset \left\{ g_M(\cdot, o) \geq (C_\nu R^{2-\nu} + 1)^{e^{1/2}} - 1 \right\} \supset B_{1+\text{diam } E}$$

provided $R > C_4$, where C_3, C_4 depend only on $\nu, I_\nu(M), \text{diam } E, \inf_{x \in B_{2+\text{diam } E}} \text{inj}(M, x), |B_{2+\text{diam } E}|$ and the infimum of the Gaussian curvature of ds^2 on $B_{2+\text{diam } E}$. Thus u is holomorphic on $B_{1+\text{diam } E}$ (note that o is a removable singularity) and the remaining argument is similar as above. \square

6. CONVERGENCE OF RIEMANNIAN SURFACES AND CONVERGENCE OF COMPLEX STRUCTURES

Let M be an orientable surface. Let $\{ds_j^2\}$ be a sequence of Riemannian metrics on M , which converges to a Riemannian metric ds^2 on M in the following sense: for each compact subset $E \subset M$ the tensor $ds_j^2 - ds^2$ and its covariant derivatives of all orders (with respect to ds^2) converge uniformly to zero on E .

Proposition 6.1. *There exist a locally finite cover $\{U_\alpha\}$ of M and holomorphic coordinates $w_j^{(\alpha)}$ (resp. $w^{(\alpha)}$) with respect to ds_j^2 (resp. ds^2) on U_α such that $w_j^\alpha - w^\alpha$ and its covariant derivatives of all order (with respect to ds^2) converge uniformly to zero on $E \cap U_\alpha$ for any compact subset $E \subset M$.*

We believe that this result is essentially known (compare [1]). For the sake of completeness, we will give a proof which relies upon the theory of elliptic operators of second order. The key ingredient is the Lax-Milgram theorem which we recall as follows. A bilinear form B on a Hilbert space H is called *bounded* if

$$|B(u, v)| \leq \text{const.} \|u\| \|v\|, \quad \forall u, v \in H$$

and *coercive* if

$$B(u, u) \geq \text{const.} \|u\|^2, \quad \forall u \in H.$$

Theorem 6.2 (Lax-Milgram, cf. [17], Theorem 5.8). *Let B be a bounded, coercive bilinear form on a Hilbert space H . For any bounded linear functional Φ on H , there exists a unique element $v = v_\Phi \in H$ such that*

$$B(u, v) = \Phi(u), \quad \forall u \in H.$$

We begin with the classical Korn-Lichtenstein procedure (see, e.g., [26], §3.11). Let ds_j^2 be given in local coordinates by

$$ds_j^2 = E_j(x, y)dx^2 + 2F_j(x, y)dxdy + G_j(x, y)dy^2$$

where $E_j G_j - F_j^2 > 0$, $E_j > 0$. For abuse of notations, we denote ds^2 by ds_∞^2 . By introducing complex coordinates $z = x + iy$, $\bar{z} = x - iy$, we can write ds_j^2 in the form

$$\sigma_j(z)|dz + \mu_j(z)d\bar{z}|^2$$

where

$$\sigma_j = \frac{E_j + G_j}{4} + \frac{1}{2}\sqrt{E_j G_j - F_j^2}, \quad \mu_j = \frac{E_j - G_j}{4\sigma_j} + i\frac{F_j}{2\sigma_j}.$$

If there is a smooth solution w_j of the following Beltrami equation

$$(6.1) \quad \frac{\partial w}{\partial \bar{z}} = \mu_j \frac{\partial w}{\partial z}$$

such that

$$\frac{\partial v_j}{\partial x} \frac{\partial u_j}{\partial y} - \frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial y} \neq 0$$

where $w_j = v_j + iu_j$, then the metric has the form

$$(6.2) \quad ds_j^2 = \frac{\sigma_j}{|\partial w_j / \partial z|^2} dw_j d\bar{w}_j,$$

so that if $(U_\alpha, w_j^{(\alpha)})$ and $(U_\beta, w_j^{(\beta)})$ are two coordinate patches with $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\partial w_j^{(\beta)} / \partial \overline{w_j^{(\alpha)}} = 0 \quad \text{on } U_\alpha \cap U_\beta,$$

i.e., M admits a complex structure given by $\{(U_\alpha, w_j^{(\alpha)})\}$, where $\{U_\alpha\}$ is a suitable cover of M . The point is that we can make the cover independent of j , in view of the convergence of ds_j^2 .

With $w_j = v_j + iu_j$, (6.1) becomes

$$(6.3) \quad \begin{aligned} \frac{\partial v_j}{\partial x} &= -\frac{F_j}{\sqrt{E_j G_j - F_j^2}} \frac{\partial u_j}{\partial x} + \frac{E_j}{\sqrt{E_j G_j - F_j^2}} \frac{\partial u_j}{\partial y} \\ \frac{\partial v_j}{\partial y} &= -\frac{G_j}{\sqrt{E_j G_j - F_j^2}} \frac{\partial u_j}{\partial x} + \frac{F_j}{\sqrt{E_j G_j - F_j^2}} \frac{\partial u_j}{\partial y} \end{aligned}$$

By using $\partial^2 v_j / \partial x \partial y = \partial^2 v_j / \partial y \partial x$, we derive that u_j satisfies the following equation

$$(6.4) \quad L_j u := a_{11}^j \frac{\partial^2 u}{\partial x^2} - 2a_{12}^j \frac{\partial^2 u}{\partial x \partial y} + a_{22}^j \frac{\partial^2 u}{\partial y^2} + b_1^j \frac{\partial u}{\partial x} + b_2^j \frac{\partial u}{\partial y} = 0$$

where

$$a_{11}^j = \frac{G_j}{\sqrt{E_j G_j - F_j^2}}, \quad a_{12}^j = \frac{F_j}{\sqrt{E_j G_j - F_j^2}}, \quad a_{22}^j = \frac{E_j}{\sqrt{E_j G_j - F_j^2}},$$

and

$$b_1^j = \frac{a_{11}^j}{\partial x} - \frac{a_{12}^j}{\partial y}, \quad b_2^j = \frac{a_{22}^j}{\partial y} - \frac{a_{12}^j}{\partial x}.$$

To solve (6.1), it suffices to find a smooth solution u_j of the second-order *elliptic* differential equation (6.4) in some (simply-connected) neighborhood of an arbitrary point z_0 whose derivatives of order one at z_0 do not vanish, for v_j may be determined by (6.3) such that

$$\frac{\partial v_j}{\partial x} \frac{\partial u_j}{\partial y} - \frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial y} = \frac{1}{E_j G_j - F_j^2} \left(G_j \left(\frac{\partial u_j}{\partial x} \right)^2 - 2F_j \frac{\partial u_j}{\partial x} \frac{\partial u_j}{\partial y} + E_j \left(\frac{\partial u_j}{\partial y} \right)^2 \right) > 0$$

holds at z_0 .

By an affine change of coordinates (which is uniform in j), we may assume that $z_0 = 0$ and

$$a_{11}^j(0) = a_{22}^j(0) = 1, \quad a_{12}^j(0) = 0.$$

Let Δ_r denote the disc in \mathbb{R}^2 with center 0 and radius r . Fix $r > 0$ for a moment. Let $C_0^\infty(\Delta_r)$ denote the set of real-valued smooth functions with compact supports in Δ_r and $L^2(\Delta_r)$ be the completion of $C_0^\infty(\Delta_r)$ with respect to the L^2 -norm $\|\cdot\|_2$ in the Lebesgue measure of \mathbb{R}^2 . We may also define Sobolev spaces $W^{k,2}$ and $W_0^{k,2}$ by standard ways.

Recall that the first eigenvalue $\lambda_1(\Delta_r)$ with respect to $-\partial^2/\partial x^2 - \partial^2/\partial y^2$ equals cr^{-2} for some numerical constant $c > 0$, so that

$$\|\nabla u\|_2^2 \geq cr^{-2}\|u\|_2^2, \quad \forall u \in W_0^{1,2}(\Delta_r)$$

We introduce a bilinear form B_j for the Hilbert space $W_0^{1,2}(\Delta_r)$ as follows

$$\begin{aligned} B_j(u, v) = & (a_{11}^j u_x, v_x) - 2(a_{12}^j u_x, v_y) + (a_{22}^j u_y, v_y) \\ & - ((b_1^j - \partial a_{11}^j/\partial x + 2\partial a_{12}^j/\partial y)u_x, v) - ((b_2^j - \partial a_{22}^j/\partial y)u_y, v). \end{aligned}$$

Clearly, we have

$$(-L_j u, u) = (B_j u, u), \quad \forall u \in C_0^\infty(\Delta_r)$$

and

$$|B_j(u, v)| \leq C\|u\|_{1,2}\|v\|_{1,2}, \quad \forall u, v \in W_0^{1,2}(\Delta_r).$$

Here and in what follows in this section we always assume that j is *sufficiently large* and C is a generic constant independent of r and j .

Furthermore, we have

$$B_j(u, u) \geq \frac{3}{4}\|\nabla u\|_2^2 - C\|u\|_2\|\nabla u\|_2 \geq \frac{1}{2}\|u\|_{1,2}^2$$

for all $u \in W_0^{1,2}(\Delta_r)$ and all $r \leq \varepsilon_0$, where ε_0 is independent of j .

We look for a smooth function ζ_j satisfying

- (1) $\partial \zeta_j/\partial x = \partial \zeta_j/\partial y = 1$ at 0;
- (2) $L_j \zeta_j$ and its derivatives of order ≤ 2 vanish at 0, i.e.,

$$(6.5) \quad |L_j \zeta_j| \leq Cr^3 \quad \text{and} \quad |\nabla(L_j \zeta_j)| \leq Cr^2.$$

Let

$$\xi_j = x + y - \frac{b_1^j(0)}{2}x^2 - \frac{b_2^j(0)}{2}y^2.$$

Clearly, we have $L_j \xi_j(0) = 0$. Put

$$\eta_j = \xi_j - \frac{1}{6} \frac{\partial L_j \xi_j}{\partial x}(0) x^3 - \frac{1}{6} \frac{\partial L_j \xi_j}{\partial y}(0) y^3.$$

It is easy to see that $L_j \eta_j$ and its derivatives of order one vanish at 0. Thus we may take

$$\begin{aligned} \zeta_j = & \eta_j - \frac{1}{24} \frac{\partial^2 L_j \eta_j}{\partial x^2}(0) x^4 - \frac{1}{24} \frac{\partial^2 L_j \eta_j}{\partial y^2}(0) y^4 \\ & - \frac{1}{12} \frac{\partial^2 L_j \eta_j}{\partial x \partial y}(0) (x^3 y + x y^3). \end{aligned}$$

By virtue of Theorem 6.2, we may find a solution $\hat{u}_j \in W_0^{1,2}(\Delta_r)$ of the equation

$$-L_j u = L_j \zeta_j,$$

which is smooth, for L_j is elliptic and $L_j \zeta_j$ is smooth, and satisfies

$$\|\hat{u}_j\|_{1,2}^2 \leq 2|B_j(\hat{u}_j, \hat{u}_j)| = 2|(\hat{u}_j, L_j \zeta_j)| \leq 2\|\hat{u}_j\|_{1,2} \|L_j \zeta_j\|_2,$$

i.e.,

$$\|\hat{u}_j\|_{1,2} \leq \sqrt{2} \|L_j \zeta_j\|_2.$$

By using dilatation $z \mapsto z/r$, we infer from Sobolev's inequality and Garding's inequality (see e.g., [17], Theorem 8.10) that

$$\begin{aligned} \sup_{\Delta_{r/4}} |D\hat{u}_j| &\leq Cr^{-2} (\|L_j \hat{u}_j\|_{1,2} + \|\hat{u}_j\|_{1,2}) \\ &\leq Cr^{-2} \|L_j \zeta_j\|_{1,2} \leq Cr \end{aligned}$$

in view of (6.5). Thus $u_j := \zeta_j + \hat{u}_j$ gives a solution of the equation (6.4), whose differential at every point of $\Delta_{r/4}$ does not vanish provided $r \leq \varepsilon_0 \ll 1$. The same is true for the corresponding isothermal parameter w_j .

Finally, we will verify the convergence of $\{w_j\}$. The argument is standard (see e.g., [30], Theorem 7.5). Fix first $r \leq \varepsilon_0$. We have

$$\|\hat{u}_j - \hat{u}\|_{1,2} \leq \sqrt{2} \|L_j(\hat{u}_j - \hat{u})\|_2 \leq \sqrt{2} (\|L_j \zeta_j - L\zeta\|_2 + \|(L_j - L)\hat{u}\|_2) \rightarrow 0$$

as $j \rightarrow \infty$. It follows again from Sobolev's inequality and Garding's inequality that for each $l \in \{0\} \cup \mathbb{Z}^+$,

$$\begin{aligned} \sup_{\Delta_{r/4}} |D^l(\hat{u}_j - \hat{u})| &\leq \text{const}_{l,r} \|\hat{u}_j - \hat{u}\|_{W^{l+2,2}(\Delta_{r/2})} \\ &\leq \text{const}_{l,r} (\|L_j(\hat{u}_j - \hat{u})\|_{l,2} + \|\hat{u}_j - \hat{u}\|_2) \\ &\leq \text{const}_{l,r} (\|L_j \zeta_j - L\zeta\|_{l,2} + \|(L_j - L)\hat{u}\|_{l,2} + \|\hat{u}_j - \hat{u}\|_2) \end{aligned}$$

where we use D^l to denote any derivative of order l . Thus $D^l(\hat{u}_j - \hat{u})$ converges uniformly to zero on $\Delta_{r/4}$. The same is true for $u_j - u$ and $v_j - v$, hence for $w_j - w$.

7. LOCAL STABILITY OF THE BERGMAN KERNEL

Let $\Omega \subset \subset \Omega' \subset \subset M$ be two open sets. Let $\{ds_j^2\}$ be a sequence of Riemannian metrics on Ω' , which converges uniformly on $\overline{\Omega}$ to a Riemannian metric ds^2 on Ω' in the following sense: the tensor $ds_j^2 - ds^2$ and its covariant derivatives of all orders (with respect to ds^2) converge uniformly to zero on $\overline{\Omega}$. By virtue of Proposition 6.1, we can choose a locally finite cover $\{U_\alpha\}$ of Ω' and holomorphic coordinates $w_j^{(\alpha)}$ (resp. $w^{(\alpha)}$) with respect to ds_j^2 (resp. ds^2) on U_α such that $w_j^\alpha - w^\alpha$ and its covariant derivatives of all order (with respect to ds^2) converge uniformly to zero on $\overline{\Omega}$.

Let $K_{D,j}$ (resp. K_D) denote the Bergman kernel of an open set $D \subset \Omega'$, with respect to the complex structure $\{(U_\alpha, w_j^{(\alpha)})\}$ (resp. $\{(U_\alpha, w^{(\alpha)})\}$).

Proposition 7.1. *For each $\varepsilon > 0$ and each compact set $E \subset \Omega$, there exists an integer j_0 such that for all $j \geq j_0$ and $z \in E$ we have*

$$\sqrt{-1}K_{\Omega,j}(z) \geq \sqrt{-1}K_{\Omega'}(z) - \varepsilon\omega, \quad \sqrt{-1}K_{\Omega}(z) \geq \sqrt{-1}K_{\Omega',j}(z) - \varepsilon\omega.$$

Here ω denotes the Kähler form of ds^2 .

Proof. Fix $z_0 \in E$ for a moment. Let f be a holomorphic differential on the Riemann surface (Ω', w) which satisfies

$$f(z_0) \wedge \overline{f(z_0)} = K_{\Omega'}(z_0), \quad \frac{\sqrt{-1}}{2} \int_{\Omega'} f \wedge \bar{f} = 1.$$

By Cauchy's estimates, we have

$$(7.1) \quad \sup_{\overline{\Omega}} \{|f|^2, |\partial f|^2\} \leq \text{const}_{\Omega, \Omega'} \left| \int_{\Omega'} f \wedge \bar{f} \right| = \text{const}_{\Omega, \Omega'},$$

where $|\cdot|$ is given with respect to ds^2 . Since

$$\frac{\partial f^*}{\partial \bar{w}_j} = \frac{\partial f^*}{\partial w} \frac{\partial w}{\partial \bar{w}_j} + \frac{\partial f^*}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial \bar{w}_j} = \frac{\partial f^*}{\partial w} \frac{\partial w}{\partial \bar{w}_j},$$

where f^* is a local representation of f , so we obtain

$$(7.2) \quad \sup_{\overline{\Omega}} |\bar{\partial}_j f|^2 \leq \text{const}_{\Omega, \Omega'} \sup_{\overline{\Omega}} \left| \frac{\partial w}{\partial \bar{w}_j} \right|^2 =: \text{const}_{\Omega, \Omega'} \varepsilon_j$$

where $\bar{\partial}_j$ denotes the Cauchy-Riemann operator for the Riemann surface (Ω', w_j) .

Since (Ω', w) is a Stein manifold in view of the Behnke-Stein theorem, it admits a smooth strictly subharmonic function φ . We claim that φ is also strictly subharmonic on $(\overline{\Omega}, w_j)$ provided j sufficiently large. To see this, simply note that

$$(7.3) \quad \begin{aligned} \frac{\partial^2 \varphi}{\partial w_j \partial \bar{w}_j} &= \left(\frac{\partial^2 \varphi}{\partial w^2} \frac{\partial w}{\partial w_j} + \frac{\partial^2 \varphi}{\partial w \partial \bar{w}} \frac{\partial \bar{w}}{\partial w_j} \right) \frac{\partial w}{\partial \bar{w}_j} + \frac{\partial \varphi}{\partial w} \frac{\partial^2 w}{\partial w_j \partial \bar{w}_j} \\ &\quad + \left(\frac{\partial^2 \varphi}{\partial w \partial \bar{w}} \frac{\partial w}{\partial w_j} + \frac{\partial^2 \varphi}{\partial \bar{w}^2} \frac{\partial \bar{w}}{\partial w_j} \right) \frac{\partial \bar{w}}{\partial \bar{w}_j} + \frac{\partial \varphi}{\partial \bar{w}} \frac{\partial^2 \bar{w}}{\partial w_j \partial \bar{w}_j} \\ &\rightarrow \frac{\partial^2 \varphi}{\partial w \partial \bar{w}} \end{aligned}$$

uniformly on $\overline{\Omega}$ as $j \rightarrow \infty$.

By Hörmander's L^2 -estimates for $\bar{\partial}$ (see e.g., [24]), there exists a smooth solution u_j of $\bar{\partial}_j u = \bar{\partial}_j f$ on Ω such that

$$(7.4) \quad \frac{\sqrt{-1}}{2} \int_{\Omega} u_j \wedge \bar{u}_j e^{-\varphi} \leq \int_{\Omega} |\bar{\partial}_j f|_{i\bar{\partial}_j \bar{\partial}_j \varphi}^2 e^{-\varphi} dV_j \leq \text{const}_{\Omega, \Omega', \varphi} \varepsilon_j$$

in view of (7.2) and (7.3) (note also that φ is bounded on Ω). Put $f_j = f - u_j$. We see that f_j is a holomorphic differential on the Riemann surface (Ω, w_j) such that

$$\|f_j\|_2 \leq \|f\|_2 + \|u_j\|_2 \leq 1 + \text{const}_{\Omega, \Omega', \varphi} \sqrt{\varepsilon_j}.$$

Now fix a positive number $r = r(\Omega, \Omega')$ such that there is a holomorphic coordinate disc $\Delta_r(z_0) \subset (\Omega, w_j)$ for all sufficiently large j . Write $u_j = u_j^* dw_j$ on $\Delta_r(z_0)$. Let χ be the cut-off function in the proof of Proposition 5.3. By the Cauchy integral formula, we have

$$\begin{aligned} u_j^*(z_0) &= \frac{1}{2\pi\sqrt{-1}} \int \frac{\bar{\partial}(\chi(|w_j|/r_0)u_j^*)/\partial \bar{w}_j}{w_j} dw_j \wedge d\bar{w}_j \\ &= \frac{1}{2\pi\sqrt{-1}} \int \frac{u_j^* \bar{\partial}\chi(|w_j|/r_0)/\partial \bar{w}_j}{w_j} dw_j \wedge d\bar{w}_j \\ &\quad + \frac{1}{2\pi\sqrt{-1}} \int \frac{\chi(|w_j|/r_0) \bar{\partial}u_j^*/\partial \bar{w}_j}{w_j} dw_j \wedge d\bar{w}_j. \end{aligned}$$

Thus

$$|u_j(z_0)|^2 \leq \text{const}_{\Omega, \Omega', \varphi} \varepsilon_j$$

in view of (7.2), (7.4). Thus

$$\sqrt{-1}K_{\Omega, j}(z_0) \geq \sqrt{-1} \frac{f_j(z_0) \wedge \overline{f_j(z_0)}}{\|f_j\|_2^2} \geq \sqrt{-1}K_{\Omega'}(z_0) - \text{const}_{\Omega, \Omega', \varphi} \sqrt{\varepsilon_j} \omega,$$

for $|K_{\Omega'}(z_0)| \leq \text{const}_{\Omega, \Omega'}$. The other inequality can be verified similarly. \square

8. PROOF OF THEOREM 1.1

Clearly the convergence $(M_j, ds_j^2, p_j) \rightarrow (M, ds^2, p)$ implies that

$$(8.1) \quad \sup_j \sup_{B_r(p_j)} |\nabla^k \text{Rm}(ds_j^2)| < \infty, \quad \forall r > 0, \quad \forall k \in \{0\} \cup \mathbb{Z}^+$$

where Rm denotes the Riemannian curvature tensor and ∇^k denotes any covariant derivative of order k .

For the sake of simplicity, we put $B_r^j = B_r(p_j)$ and $B_r = B_r(p)$. Let $E \subset M$ be a compact set with $p \in E$. Put $E_j = \phi_j(E)$. By virtue of (8.1), (5.7), (5.8) and (5.9), there exists a positive constant R_0 such that for all $R \geq R_0$,

$$(8.2) \quad \sup_{z \in E_j} |K_{M_j}(z) - K_{B_R^j}(z)|_{ds_j^2} \leq C |\log R|^{-1}.$$

Here and in what follows in this section we always assume that j is *sufficiently large*, and C is a generic constant independent of j and R .

Now fix an arbitrary number $0 < \varepsilon < |\log R_0|^{-1}$ for a moment. Put $R = e^{1/\varepsilon}$. It is easy to see that

$$B_{R-1} \subset \Omega_R^j := \phi_j^{-1}(B_R^j) \subset B_{R+1}.$$

By (8.2), we have

$$(8.3) \quad \sup_{z \in E} |\phi_j^* K_{M_j}(z) - \phi_j^* K_{B_R^j}(z)|_{\phi_j^*(ds_j^2)} \leq C\varepsilon.$$

The point is that $\phi_j^* K_{B_R^j}$ actually coincides with $K_{\Omega_R^j, j}$, the Bergman kernel of Ω_R^j with respect to the complex structure induced by $\phi_j^*(ds_j^2)$. Thus (8.3) yields

$$(8.4) \quad \sqrt{-1}\phi_j^* K_{M_j}(z) \geq \sqrt{-1}K_{\Omega_R^j, j}(z) - C\varepsilon\omega, \quad \forall z \in E.$$

Since $\phi_j^*(ds_j^2) \rightarrow ds^2$ uniformly on B_{R+2} in the sense of § 7, we have

$$(8.5) \quad \begin{aligned} \sqrt{-1}K_{\Omega_R^j, j}(z) &\geq \sqrt{-1}K_{B_{R+1}, j}(z) \geq \sqrt{-1}K_{B_{R+2}}(z) - \varepsilon\omega \\ &\geq \sqrt{-1}K_M(z) - \varepsilon\omega \end{aligned}$$

in view of Proposition 7.1. Thus

$$\sqrt{-1}\phi_j^* K_{M_j}(z) \geq \sqrt{-1}K_M(z) - C\varepsilon\omega, \quad \forall z \in E$$

in view of (8.4) and (8.5). It follows again from Proposition 7.1 that

$$\sqrt{-1}K_{\Omega_R^j, j}(z) \leq \sqrt{-1}K_{B_{R-1}, j}(z) \leq \sqrt{-1}K_{B_{R-2}}(z) + \varepsilon\omega.$$

Together with (8.3), we obtain

$$\sqrt{-1}\phi_j^* K_{M_j}(z) \leq \sqrt{-1}K_{B_{R-2}}(z) + C\varepsilon\omega.$$

A normal family argument shows

$$K_{B_{R-2}}(z) \rightarrow K_M(z)$$

uniformly on E as $R \rightarrow \infty$. Thus we have verified that $\phi_j^* K_{M_j}(z) \rightarrow K_M(z)$. The convergence of their covariant derivatives can be verified by using the Cauchy integral formula. We leave the details to the reader.

9. PROOF OF THEOREM 1.2

Let E be a compact set in M which contains p and $E_1 := \{z \in M : \text{dist}(z, E) \leq 1\}$. Let $\{h_k\}$ be a complete orthonormal basis of $\mathcal{H}(M)$. Let $K_M(z, w)$ denote the off-diagonal Bergman kernel, i.e.,

$$K_M(z, w) = \sum_k h_k(z) \otimes \overline{h_k(w)}.$$

We define $|K_M(z, w)|$ as follows. Let $ds^2 = \lambda dw d\bar{w}$ (resp. $\mu dz d\bar{z}$) at w (resp. z). If we write $K_M(z, w) = K_M^*(z, w) dz \otimes d\bar{w}$, then

$$|K_M(z, w)|^2 := \frac{|K_M^*(z, w)|^2}{\lambda(w)\mu(z)}.$$

Fubini's theorem yields

$$\int_{M \times E_1} |K_M(z, w)|^2 = \int_{E_1} |K_M(w, w)|^2 < \infty,$$

so that for any $\varepsilon > 0$,

$$\int_{(M \setminus B_R(p)) \times E_1} |K_M(z, w)|^2 < \varepsilon$$

provided R sufficiently large. Applying the sub-mean-value inequality as the proof of Proposition 5.3, we obtain

$$|K_M(z, w)|^2 \leq \text{const.} \int_{\zeta \in E_1} |K_M(z, \zeta)|^2$$

for all $z \in M$ and $w \in E$, where the constant depends only on E . Thus

$$\int_{z \in M \setminus B_R(p)} |K_M(z, w)|^2 \leq \text{const.} \varepsilon.$$

Put

$$h(z, w) = \sum_k \overline{h_k^*(w)} h_k(z)$$

where h_k^* is a local representation of h_k at w . Then we have

$$\lambda(w)^{-1} \int_{z \in M \setminus B_R(p)} |h(z, w)|^2 \leq \text{const.} \varepsilon.$$

Now fix $w \in E$ and R for a moment. Clearly, we have $R_j > 2R$ for $j \gg 1$. Let χ be the cut-off function in the proof of Proposition 5.3 and let ρ denote the distance from p on M . By the proofs of Propositions 5.3, 5.4, we may write

$$\chi(\rho/R_j)h(\cdot, w) = \tilde{h}_j \oplus u_j,$$

where $\tilde{h}_j \in \mathcal{H}(M_j)$ and u_j satisfies

$$\bar{\partial}u_j = \bar{\partial}\chi(\rho/R_j) \wedge h(\cdot, w) =: v_j$$

and

$$\begin{aligned}
\int_{M_j} |u_j|^2 &\leq \frac{4}{\lambda_1(M_j)} \int_{M_j} |v_j|^2 \\
&\leq \frac{C_0}{\lambda_1(M_j) R_j^2} \int_{M \setminus B_{\frac{1}{2}R_j}(p)} |h(\cdot, w)|^2 \\
&\leq \text{const.} \lambda(w) \varepsilon
\end{aligned}$$

where C_0 is a universal constant. Again by the sub-mean-value inequality,

$$|u_j(w)|^2 \leq \text{const.} \lambda(w) \varepsilon,$$

i.e., $|u_j^*(w)|^2 \leq \text{const.} \lambda(w)^2 \varepsilon$. Note that

$$\begin{aligned}
\|\tilde{h}_j\|_{L^2(M_j)} &\leq \|h(\cdot, w)\|_{L^2(M)} + \|u_j\|_{L^2(M_j)} \\
&\leq K_M^*(w, w)^{1/2} + O(\sqrt{\lambda(w) \varepsilon}).
\end{aligned}$$

It follows that

$$K_{M_j}^*(w, w) \geq \frac{K_M^*(w, w)^2 - O(\lambda(w)^2 \varepsilon)}{(K_M^*(w, w)^{1/2} + O(\sqrt{\lambda(w) \varepsilon}))^2},$$

i.e.,

$$|K_{M_j}(w, w)| \geq \frac{|K_M(w, w)|^2 - O(\varepsilon)}{(|K_M(w, w)|^{1/2} + O(\sqrt{\varepsilon}))^2} \geq |K_M(w, w)| - O(\sqrt{\varepsilon}).$$

On the other side, we have

$$|K_{M_j}(w, w)| \leq |K_{B_{R_j}(p)}(w, w)| \rightarrow |K_M(w, w)|$$

as $j \rightarrow \infty$. The proof is complete.

10. CONCLUDING REMARKS

1. In general, the Cheeger-Gromov convergence does not imply the convergence of the Bergman kernel.

Proposition 10.1. *Let $\{(M_j, ds_j^2)\}$ be a sequence of compact Riemannian surfaces which converges to a complete Riemannian surface (M, ds^2) such that*

- (1) $\dim \mathcal{H}(M) = \infty$,
- (2) $K_{M_j} \rightarrow K_M$ in the sense of Theorem 1.1.

Then the genus g_j of M_j tends to infinity as $j \rightarrow \infty$.

Proof. Take first a complete orthonormal basis $h_{j,1}, \dots, h_{j,g_j}$ of $\mathcal{H}(M_j)$. Note that

$$K_{M_j}(z) = \sum_{k=1}^{g_j} h_{j,k}(z) \wedge \overline{h_{j,k}(z)}.$$

Thus

$$\begin{aligned}
g_j = \dim_{\mathbb{C}}(\mathcal{H}(M_j)) &= \frac{\sqrt{-1}}{2} \int_{M_j} K_{M_j}(z) \geq \frac{\sqrt{-1}}{2} \int_{\phi_j(\Omega_j)} K_{M_j}(z) \\
&= \frac{\sqrt{-1}}{2} \int_{\Omega_j} \phi_j^* K_{M_j}
\end{aligned}$$

where ϕ_j and Ω_j are given as Definition 1.1. Since $\phi^*K_{M_j} \rightarrow K_M$ locally uniformly on M , so we have

$$\liminf_{j \rightarrow \infty} g_j \geq \frac{\sqrt{-1}}{2} \int_E K_M(z)$$

for any compact set $E \subset M$. It follows immediately that

$$\liminf_{j \rightarrow \infty} g_j \geq \frac{\sqrt{-1}}{2} \int_M K_M(z) = \infty.$$

□

Example. Let M be a semi-sphere in the sphere $S^2 \subset \mathbb{R}^3$, which is conformally equivalent to the Poincaré disc (Δ, ds^2) by Riemann's mapping theorem. Let $\{\Omega_j\}$ be a sequence of precompact open subsets exhausting M . Let $M_j = (S^2, ds_j^2)$ where ds_j^2 is a Riemannian metric with $ds_j^2 = ds^2$ on Ω_j . Thus $(M_j, ds_j^2) \rightarrow (M, ds^2)$, whereas $K_{M_j} \nrightarrow K_M$.

2. The case of convergent hyperbolic surfaces is of independent interest. Let us recall the following

Definition 10.1 (cf. [38]). A sequence $\{\Gamma_j\}$ of closed subgroups of a Lie group converges geometrically to a group Γ if

- (1) each $\gamma \in \Gamma$ is the limit of a sequence $\{\gamma_j\}$, with $\gamma_j \in \Gamma_j$,
- (2) the limit of every convergent sequence $\{\gamma_{j_k}\}$, with $\gamma_{j_k} \in \Gamma_{j_k}$ is in Γ .

It is known that if a sequence $\{\Gamma_j\}$ of torsion-free Fuchsian groups converges geometrically to a non-elementary Fuchsian group Γ , then $\mathbb{D}/\Gamma_j \rightarrow \mathbb{D}/\Gamma$ in the sense of Cheeger-Gromov, where \mathbb{D} denotes the unit disc (see e.g., [33], Theorem 7.6). Thus by Theorem 1.1 and (3.6), we immediately get the following

Proposition 10.2. Let $\{\Gamma_j\}$ be a sequence of torsion-free Fuchsian groups converges geometrically to a non-elementary Fuchsian group Γ and satisfies $\sup_j \delta(\Gamma_j) < 1$. Then $K_{\mathbb{D}/\Gamma_j} \rightarrow K_{\mathbb{D}/\Gamma}$.

3. Following [6], we may construct a family $\{M_t\}$ of compact Riemannian surfaces with $\lambda_1(M_t) \geq \text{const.} > 0$ in the following way:

Let M be a hyperbolic Riemannian surface with $2n$ cusps $\{C_i\}$. Let M_t be the surface formed from M by first replacing the punctures w.r.t. C_i with geodesics of length t and then gluing the geodesic w.r.t. C_{2i-1} with the geodesic w.r.t. C_{2i} . One may perturb the hyperbolic metric on M slightly to obtain a new Riemannian metric on M_t , so that $M_t \rightarrow M$ in the sense of Cheeger-Gromov, and $\lambda_1(M_t) \geq \text{const.} > 0$ as $t \rightarrow 0$.

Another interesting way is to consider a family of non-singular connected levels $M_t = f^{-1}(t)$, $t \in \mathbb{R}$, of a polynomial f on the Euclidean sphere S^3 , when M_t approaches the singular surface M_s (w.r.t. the Euclidean metric) as $t \rightarrow s$. It is known that if M_s is irreducible and $\dim V_s = \dim V_t$ then $\lambda_1(M_t) \geq \text{const.} > 0$ as $t \rightarrow s$ (see Gromov [22], p. 252).

4. A conic degenerating family $\{M_t\}$ of compact Riemannian surfaces has $I_2(M_t) \geq \text{const.} > 0$ and $|M_t| \leq \text{const.}$, provide that the pinching geodesic is nonseparating (see [25], Corollary 2.9 and the proof of Proposition 2.6). It follows that

$$I_2(M_t)|M_t|^{-1/2} \geq \text{const.} > 0.$$

Effective convergence the Bergman kernel and related invariants for some special degenerating analytic families of compact Riemannian surfaces was established in [44], [25].

5. Let M_∞ be a complete Riemannian manifold such that there exists a nested sequence of torsion-free discrete groups of isometries

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \cap \Gamma_j = \{\text{id}\}.$$

Let $M_j = M_\infty/\Gamma_j$ be endowed with the metric induced by the complete metric on M_∞ . Then $M_j \rightarrow M_\infty$ in the sense of Cheeger-Gromov (we may choose $\phi_j = \pi_j|_{\mathcal{D}_j}$ where $\pi_j : M_\infty \rightarrow M_j$ is the covering map and \mathcal{D}_j is suitable fundamental region). After the seminal work of Kazhdan (see [28], [29]), stability properties of the Bergman kernel when all M_j are complex manifolds were studied extensively (cf. [35], [15], [42], [34], [11], [43]). In particular, Rhodes [35] proved $K_{M_j} \rightarrow K_M$ for Riemannian surfaces satisfying $\lambda_1(M_j) \geq \text{const.} > 0$, whereas Ohsawa [34] gave a counterexample for general case.

6. Below we provide a sequence of compact Riemannian surfaces which satisfies the conditions of Theorem 1.2, whereas Theorem 1.1 does not apply. Let us start from a compact Riemann surface M_0 with genus ≥ 2 . Let M be a regular covering of M_0 whose deck transformation group Γ is isomorphic with \mathbb{Z} . For instance, we may choose M to be a Schottky type covering of M_0 by first taking a ring cut γ of M_0 then connecting infinitely many copies of $M_0 \setminus \gamma$ along the opposite shores of γ (see e.g. [40], Chapter X, § 14).

Put $\Gamma = \{g_k : k \in \mathbb{Z}\}$. Let M_j be a compact Riemannian surface obtained by adding a spherical cap \mathcal{C} to each end of the set

$$\bigcup_{\{k \in \mathbb{Z} : |k| \leq j\}} g_k(M_0 \setminus \gamma).$$

We may introduce a Riemannian metric on M_j by patching up together the metric on M and the Euclidean metric on the spherical cap. Clearly there exists $R_j \approx j$ such that $B_{R_j}(p) \subset M_j$ for some fixed point $p \in M$. Since M has bounded geometry, so we have the following isoperimetric inequality

$$(10.1) \quad |\partial\Omega| \geq \text{const.} \min \left\{ 1, \sqrt{|\Omega|} \right\}$$

(see [19], Theorem 7.7). We claim that

$$(10.2) \quad I_\infty(M_j) \geq \text{const.} j^{-1}.$$

To see this, let S be a smooth hypersurface that divides M_j into two disjoint open subsets Ω_1, Ω_2 . According to Yau [41], it suffices to consider the case when both Ω_1 and Ω_2 are connected. Suppose $|\Omega_1| \leq |M_j|/2$. If $\Omega_1 \subset M$, we infer from (10.1) that

$$\frac{|\partial\Omega_1|}{|\Omega_1|} \geq \text{const.} |\Omega_1|^{-1} \geq \text{const.} |M_j|^{-1} \geq \text{const.} j^{-1}$$

when $|\Omega_1| \geq 1$, and

$$(10.3) \quad \frac{|\partial\Omega_1|}{|\Omega_1|} \geq \text{const.}$$

when $|\Omega_1| \leq 1$. If Ω_1 is contained in a spherical cap slightly larger than \mathcal{C} , then we still have (10.3) in view of the classical isoperimetric inequality in \mathbb{R}^2 . In the remaining case, we put $\Omega'_1 = \Omega_1 \cap M$ and $\Omega''_1 = \Omega_1 \setminus \overline{\Omega'_1}$. Then we have

$$|\partial\Omega_1| \geq \text{const.} \max\{\partial\Omega'_1, \partial\Omega''_1\},$$

so that

$$\frac{|\partial\Omega_1|}{|\Omega_1|} \geq \text{const.} \min \left\{ \frac{|\partial\Omega'_1|}{|\Omega'_1|}, \frac{|\partial\Omega''_1|}{|\Omega''_1|} \right\} \geq \text{const.} j^{-1}.$$

Thus we have verified (10.2). Finally, Cheeger's inequality yields

$$\lambda_1(M_j) \geq \text{const. } j^{-2} \approx R_j^{-2}.$$

The same argument probably works when M is a \mathbb{Z}^m covering of M_0 for arbitrary $m \in \mathbb{Z}^+$.

7. We end this section by proposing the following

Problem 2. *Let $\{f_j\}$ be a sequence of smooth functions in \mathbb{R}^3 which converges locally uniformly to a smooth function f . Suppose $M_j := \{f_j = 0\}$ and $M := \{f = 0\}$ are non-singular. With respect to the complex structure induced by the Euclidean metric, when does K_{M_j} converge to K_M in some sense?*

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